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Gauge transformations in relativistic two-particle constraint theory

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Abstract

Using connection with quantum field theory, the infinitesimal covariant abelian gauge transformation laws of relativistic two-particle constraint theory wave functions and potentials are established and weak invariance of the corresponding wave equations shown. Because of the three-dimensional projection operation, these transformation laws are interaction dependent. Simplifications occur for local potentials, which result, in each formal order of perturbation theory, from the infra-red leading effects of multiphoton exchange diagrams. In this case, the finite gauge transformation can explicitly be represented, with a suitable approximation and up to a multiplicative factor, by a momentum dependent unitary operator that acts in x-space as a local dilatation operator. The latter is utilized to reconstruct from the Feynman gauge the potentials in other linear covariant gauges. The resulting effective potential of the final Pauli-Schrödinger type eigenvalue equation has the gauge invariant attractive singularity α^2/r^2, leading to a gauge invariant critical coupling constant $\alpha_c = 1/2$.

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1 Introduction

The knowledge of the behavior of Green's functions under gauge transformations in QED [1, 2, 3] plays a crucial role in proving gauge invariance of observables and in particular of bound state energies [4, 5]. In practical calculations, however, one generally uses approximations to exact equations, which necessitate a close control of the degree of realization of the various symmetries of the system under study.

In QED, it has appeared that the Coulomb gauge is the most convenient gauge for treating the bound state problem, since it allows the optimal expansion of the Bethe-Salpeter equation [6, 7, 8] around the nonrelativistic theory [4, 9, 10, 11, 12]. Covariant gauges produce, at each formal order of perturbation theory, spurious infra-red singularities that are cancelled only by higher order diagrams [13, 14] and thus become of less practical interest. The main disadvantage of the Coulomb gauge is, however, its noncovariant nature, which does not allow its incorporation in covariant equations.

From this viewpoint, constraint theory, which leads to a manifestly covariant three-dimensional description of two-body systems [15, 16, 17], has opened a new perspective in the subject. It was shown [18, 19, 20] that the expansion of the Bethe-Salpeter equation around the constraint theory wave equations in the Feynman gauge (as well as for scalar interactions) is free of the abovementioned diseases of covariant gauges and allows a systematic study of infra-red leading effects of multiphoton exchange diagrams; the latter can then be represented in three-dimensional x-space as local potentials. Summing the series of these leading terms one obtains a local potential in compact form [20], which is well suited for a continuation to the strong coupling domain of the theory or for a generalization to other effective interactions.

The purpose of the present paper is to investigate the forms of the local potentials in linear covariant gauges and to find the relationships between them. Our approach is accomplished in two steps. First, using the connection of the constraint theory wave equations with the Bethe-Salpeter equation, we determine the infinitesimal gauge transformation laws of constraint theory wave functions and potentials for fermion-antifermion systems and establish the (weak) invariance of the corresponding wave equations. Second, specializing to the local potential approximation, we show that the finite gauge transformation can be represented, with a suitable approximation and up to a multiplicative factor, by a momentum dependent unitary operator that acts in x-space as a local di-

latation operator. Under the action of the latter, the potentials undergo two kinds of modification. In the first one, they are changed in a form invariant way, by the replacement of their argument r (the c.m. interparticle distance) by a function $r(\xi)$, where ξ is the gauge variation parameter. Apart from a rapid variation near the origin, $r(\xi)$ is essentially dominated by its large-distance behavior, in which r is simply shifted by a constant value. In the second kind of modification, certain parts of the spacelike components of the electromagnetic potential are functionally modified. The combination of these two types of modification allows, in particular, the reconstruction of the potentials in linear covariant gauges from the knowledge of the potential in the Feynman gauge.

Studying the short-distance behavior of these (three-dimensional) potentials we find that the effective potential that appears in the final Pauli-Schrödinger type eigenvalue equation has the gauge-invariant attractive singularity α^2/r^2. As is known, to such a singularity there corresponds a critical value α_c of the coupling constant, $\alpha_c = 1/2$, at which value the fall to the center phenomenon occurs.

Constraint theory thus provides us with a three-dimensional framework in which the local potential approximation consistently fulfils the requirement of gauge invariance of the theory. These results, while deduced in the framework of QED, might also survive, with appropriate adaptations, to the incorporation or consideration of other types of interaction.

The plan of the paper is the following. In Sec. 2, the gauge transformation properties of the Green's functions and of the Bethe-Salpeter wave function are reviewed. In Sec. 3, the infinitesimal gauge transformation laws of the constraint theory wave functions and potentials are determined. The case of the local approximation of potentials is considered in Sec. 4. The general properties of the gauge transformations in the local approximation are studied in Sec. 5. A summary and concluding remarks follow in Sec. 6.

2 Gauge transformations of Green's functions and wave functions

Gauge transformation laws of Green's functions in QED can be obtained with several equivalent methods: i) by considering the operator changes in the field operators [1]; ii) by modifying in the functional integral the gauge fixing condition [2]; iii) by using the Ward-Takahashi identities [3]. In the present work we shall consider only linear covariant gauges, characterized by a parameter ξ; the photon propagator is then:

$$D_{\mu\nu}(k) \ = \ -(g_{\mu\nu} - \xi\frac{k_\mu k_\nu}{k^2})\frac{i}{k^2 + i\epsilon} \ . \tag{2.1}$$

The reference gauge is taken here to be the Feynman gauge and the transformations express a Green's function calculated in the gauge ξ with respect to its value in the Feynman gauge ($\xi = 0$). More generally, these transformations concern the passage from a gauge ξ_1 to a gauge ξ_2, with $\xi = \xi_2 - \xi_1$.

The transformation law of the unrenormalized Green's function of a charged particle (a boson or a fermion) is:

$$G_\xi(x) \ = \ \exp\{ie^2\xi(\Delta(x) - \Delta(0))\}\ G(x) \ , \tag{2.2}$$

where e is the unrenormalized electric charge of the particle and Δ is defined as:

$$\Delta(x) \ = \ \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik.x}}{(k^2 + i\epsilon)^2} \ , \qquad \Delta(k) \ = \ \frac{1}{(k^2 + i\epsilon)^2} \ . \tag{2.3}$$

Notice that the transformation law is spin independendent.

For a two-particle Green's function, with particle 1 representing a fermion (boson) with charge e_1 and particle 2 an antifermion (boson) with charge $-e_2$, the transformation law is [3, 13]:

$$\begin{aligned} G_\xi(x_1, x_2; y_1, y_2) = \exp\{i\xi[e_1^2(\Delta(x_1 - y_1) - \Delta(0)) + e_2^2(\Delta(x_2 - y_2)) - \Delta(0))]\} \\ \times \exp\{i\xi e_1 e_2[\Delta(x_1 - x_2) + \Delta(y_1 - y_2) - \Delta(x_1 - y_2) - \Delta(y_1 - x_2)]\} \\ \times G(x_1, x_2; y_1, y_2) \ . \end{aligned} \tag{2.4}$$

One also deduces from Eq. (2.4) the gauge transformation law of the Bethe-Salpeter wave function. We assume that the two-particle system is neutral in charge, hence:

$$e_1 \ = \ e_2 \ = \ e \ . \tag{2.5}$$

The time separation between outgoing and ingoing states is defined as $T = ((x_1^0 + x_2^0) - (y_1^0 + y_2^0))/2$. For large time separations ($T \to \infty$) and finite values of $x_1^0 - x_2^0$ and $y_1^0 - y_2^0$, G satisfies the cluster decomposition [7]:

$$G(x_1, x_2; y_1, y_2) \underset{T\to\infty}{=} \eta \sum_n \Phi_n(x_1, x_2)\overline{\Phi}_n(y_1, y_2) \ , \tag{2.6}$$

where $\eta = +1$ for bosonic fields and $\eta = -1$ for fermionic fields; the sum over n corresponds to a complete set of states; Φ_n is a generalized Bethe-Salpeter wave function with respect to the intermediate state $|n>$ and becomes the usual Bethe-Salpeter wave function when $|n>$ is a bound state [6, 7]. Although the function Δ has a logarithmic increase at infinity, the charge neutrality condition (2.5) leads to a mutual cancellation of singular terms and one obtains [13]:

$$\Phi_\xi(x_1, x_2) \ = \ e^{ie^2\xi(\Delta(x_1 - x_2) - \Delta(0))} \ \Phi(x_1, x_2) \ . \tag{2.7}$$

The infinite factor $\Delta(0)$ in Eqs. (2.2), (2.4) and (2.7) determines the transformation law of the wave function renormalization constant Z_2 [2]. *In the remaining part of this work, we shall not consider radiative corrections and hence shall ignore gauge transformation effects coming from photons associated entirely with one particle; only exchanged photon contributions will be taken into account.* [More generally, one may consider different gauges for exchanged photons and for photons entering in radiative corrections [4].] Except in particular limiting procedures, such as in Eq. (2.6), there are no, in general, cancellations between the two kinds of contributions, since they are concerned with different variables.

While the behavior under gauge transformations of Green's functions and wave functions is very simple (in x-space), the same is not true for the inverses of Green's functions, the vertex functions and the scattering amplitudes. In this case, only infinitesimal gauge transformations have relatively simple forms. They can be obtained either by inverting the Green's functions or by using the Ward-Takahashi identities.

The infinitesimal change of the inverse of the four-point Green's function is:

$$\delta_\xi G^{-1} \ = \ -G^{-1}(\delta_\xi G)G^{-1} \ . \tag{2.8}$$

Equations (2.8) and (2.4) can then be used to show the invariance, in a weak form, of the Bethe-Salpeter equation. The latter can directly be written in terms of G^{-1} as:

$$G^{-1}\Phi \ = \ 0 \ . \tag{2.9}$$

Invariance of this equation under gauge transformations requires:

$$(\delta_\xi G^{-1})\Phi + G^{-1}\delta_\xi\Phi \ = \ G^{-1}\Big(-(\delta_\xi G)G^{-1}\Phi + \delta_\xi\Phi\Big) \ = \ 0 \ . \tag{2.10}$$

Replacing in this equation $\delta_\xi G$ and $\delta_\xi \Phi$ by their expressions (2.4) and (2.7), respectively, one obtains:

$$\begin{aligned}
&\Delta(x_1-x_2)\int d^4z_1d^4z_2G^{-1}(x_1,x_2;z_1,z_2)\Phi(z_1,z_2)\\
&-\int d^4x_1'd^4x_2'd^4y_1d^4y_2d^4z_1d^4z_2G^{-1}(x_1,x_2;x_1',x_2')(\Delta(x_1'-y_2)+\Delta(x_2'-y_1))\\
&\times G(x_1',x_2';y_1,y_2)G^{-1}(y_1,y_2;z_1,z_2)\ \Phi(z_1,z_2) \ = \ 0 \ .
\end{aligned} \tag{2.11}$$

(The contribution of $\delta_\xi\Phi$ has been cancelled by one of the terms of $\delta_\xi G$.) The first term vanishes on account of Eq. (2.9). In the second and third terms, the gauge propagators Δ acting multiplicatively on G, join one of the x''s to the y of the other line; in momentum space, they produce crossed diagrams with G; these do not have bound state poles at the positions of the direct diagram poles; therefore, the two products ΔG cannot prevent the factor $G^{-1}\Phi$ from vanishing, on account of Eq. (2.9). One thus establishes the weak gauge invariance of the Bethe-Salpeter equation.

This conclusion is not changed when one includes radiative corrections relative to fermion or boson lines and vertices. In particular, radiative corrections that join x' to y on the same line, imply in momentum space integrations on the bound state pole position, which is then transformed into a cut.

To display the gauge transformation property of the off-mass shell scattering amplitude, we pass into momentum space. The infinitesimal form of Eq. (2.4) is:

$$\begin{aligned}
\delta_\xi G(p_1,-p_2;p_1',-p_2') \ &= \ ie^2\delta\xi\int\frac{d^4k}{(2\pi)^4}\Delta(k)\Big[G(p_1-k,-(p_2+k);p_1',-p_2')\\
&+G(p_1,-p_2,p_1'+k,-(p_2'-k))-G(p_1-k,-p_2;p_1',-(p_2'-k))\\
&-G(p_1,-(p_2+k);p_1'+k,-p_2')\Big] \ .
\end{aligned} \tag{2.12}$$

This equation is graphically represented in Fig. 1. We notice that if G has a (simple) bound state pole, then in the right-hand side of Eq. (2.12) the first two terms have also the same pole. However, this pole being simple, one concludes that the gauge transformation does not change the bound state energy [4, 5]. (Otherwise, the singularity structure in the right-hand side of Eq. (2.12) would be that of a double pole.)

The scattering amplitude T is defined from the Green's function G as:

$$T \ = \ G_1^{-1} G_2^{-1} [G - G_0] G_1^{-1} G_2^{-1} \ , \tag{2.13}$$

where G_1 and G_2 are the external particle propagators and G_0 is their product:

$$G_0 \ = \ G_1 G_2 \ . \tag{2.14}$$

The infinitesimal gauge transformation law of T is then:

$$\begin{aligned}
\delta_\xi T = ie^2 \delta\xi \Big[G_1^{-1}(p_1) - G_1^{-1}(p_1') \Big] \Big[G_2^{-1}(-p_2) - G_2^{-1}(-p_2') \Big] \Delta(p_1 - p_1') \\
+ ie^2 \delta\xi \int \frac{d^4k}{(2\pi)^4} \ \Delta(k) \ \Big[\ G_1^{-1}(p_1) G_1(p_1 - k) T(p_1 - k, -(p_2 + k); p_1', -p_2') \\
\times G_2(-(p_2 + k)) G_2^{-1}(-p_2) \\
+ G_2^{-1}(-p_2') G_2(-(p_2' - k)) T(p_1, -p_2; p_1' + k, -(p_2' - k)) G_1(p_1' + k) G_1^{-1}(p_1') \\
- G_1^{-1}(p_1) G_2^{-1}(-p_2') G_1(p_1 - k) G_2(-(p_2' - k)) T(p_1 - k, -p_2; p_1', -(p_2' - k)) \\
- T(p_1, -(p_2 + k); p_1' + k, -p_2') G_1(p_1' + k) G_2(-(p_2 + k)) G_1^{-1}(p_1') G_2^{-1}(-p_2) \ \Big] \ .
\end{aligned} \tag{2.15}$$

Taking the external particles on their mass-shell, one immediately deduces from this equation gauge invariance of the on-mass shell scattering amplitude.

The unitarity of the gauge transformation can be checked with the invariance of the norm of the Bethe-Salpeter wave function. We first notice that, according to Eq. (2.4), the adjoint $\overline{\Phi}$ of the Bethe-Salpeter wave function [Eq. (2.6)] transforms in the same way as Φ [Eq. (2.7)] (with the same sign in the exponential function); this is due to the fact that the gauge function Δ [Eq. (2.3)] is imaginary for euclidean variables in x-space. The norm of the internal part of the Bethe-Salpeter wave function is [8]:

$$(\phi, \phi) \ = \ \int d^4x \ \overline{\phi} \ \frac{\partial G^{-1}}{\partial s} \ \phi \ = \ -i\eta \ , \tag{2.16}$$

where $s = P^2 = (p_1 + p_2)^2$ and η was defined after Eq. (2.6).

Using infinitesimal gauge transformations together with Eqs. (2.7), (2.8) and (2.12) and restricting ourselves to hermitian kernels and time-reversal invariant interactions, we find (in compact notation):

$$
\begin{aligned}
\delta_\xi(\phi,\phi) = ie^2\delta\xi\Big\{ & \int \overline{\phi}\Delta\ \frac{\partial G^{-1}}{\partial s}\ \phi + \int \overline{\phi}\ \frac{\partial G^{-1}}{\partial s}\ \Delta\phi \\
& - \int \overline{\phi}\ \frac{\partial G^{-1}}{\partial s}\ \Big[(\Delta G) + (G\Delta) - (\Delta G)_{cr} - (G\Delta)_{cr}\Big]\ G^{-1}\phi \\
& - \int \overline{\phi}G^{-1}\ \Big[(\Delta G) + (G\Delta) - (\Delta G)_{cr} - (G\Delta)_{cr}\Big]\ \frac{\partial G^{-1}}{\partial s}\ \phi \\
& - \int \overline{\phi}G^{-1}\ \frac{\partial}{\partial s}\Big[(\Delta G) + (G\Delta) - (\Delta G)_{cr} - (G\Delta)_{cr}\Big]\ G^{-1}\phi\Big\}\ ,
\end{aligned}
\tag{2.17}
$$

where the subscript "cr" designates the terms corresponding to the crossed diagrams [Eq. (2.12) and Fig. 1]. In the above equation, one is entitled to use the wave equation (2.9), or its adjoint, $\overline{\phi}G^{-1} = 0$, as long as G^{-1} is not multiplied by terms having a pole at the bound state position in the s-channel. Therefore, the crossed diagram terms disappear and after some algebraic cancellations one remains with the following terms, which also ultimately vanish:

$$
\begin{aligned}
\delta_\xi(\phi,\phi) &= -ie^2\delta\xi\Big\{\int \overline{\phi}\ \frac{\partial G^{-1}}{\partial s}\ (G\Delta)G^{-1}\phi + \int \overline{\phi}G^{-1}(\Delta G)\ \frac{\partial G^{-1}}{\partial s}\ \phi \\
&\qquad + \int \overline{\phi}G^{-1}\frac{\partial}{\partial s}\Big[\Delta G + G\Delta\Big]G^{-1}\phi\Big\} \\
&= -ie^2\delta\xi\Big\{\int \overline{\phi}G^{-1}\frac{\partial}{\partial s}\Big(\Delta G G^{-1}\Big)\phi + \int \overline{\phi}\frac{\partial}{\partial s}\Big(G^{-1}G\Delta\Big)G^{-1}\phi\Big\} \\
&= -ie^2\delta\xi\Big\{\int \overline{\phi}\Big(\frac{\partial \Delta}{\partial s}\Big)G^{-1}\phi + \int \overline{\phi}G^{-1}\Big(\frac{\partial \Delta}{\partial s}\Big)\phi\Big\} \\
&= 0\ .
\end{aligned}
\tag{2.18}
$$

Let us finally remark that the gauge propagator Δ, having a singular infra-red behavior, may lead, during the evaluation of Feynman diagram integrals, to infra-red divergences. However, the integrals involved in the variations $\delta_\xi G$ [Eq. (2.12)] and $\delta_\xi T$ [Eq. (2.15)] are actually globally infra-red regular, although individual terms are divergent, as can be checked by taking in the integrands the limit $k \to 0$. In later calculations, some of the terms present in $\delta_\xi G$ or $\delta_\xi T$ will be dropped, for they do not contribute to the bound state poles. It should be understood, in such cases, that their infra-red divergent parts are kept, in order to maintain the regularity of the remaining integral.

3 Gauge transformations of constraint theory wave functions and potentials

Constraint theory [15, 16, 17] allows the elimination of the relative energy variable of the two particles by means of a manifestly covariant equation. The choice of the latter is not unique and generally various choices are related one to the other by canonical or wave function transformations. We choose the following constraint:

$$[\ (p_1^2 - p_2^2) - (m_1^2 - m_2^2)\ ]\ \widetilde{\Psi}(x_1, x_2)\ =\ 0\ , \tag{3.1}$$

where p_1 and p_2 are the momentum operators of particles 1 and 2, m_1 and m_2 their respective masses and $\widetilde{\Psi}$ is the constraint theory wave function. We use standard definitions for the total and relative variables: $P = p_1 + p_2$, $p = (p_1 - p_2)/2$, $x = x_1 - x_2$, $M = m_1 + m_2$. For states that are eigenstates of the total momentum P, we define transverse and longitudinal components of four-vectors with respect to P and denote them with indices T and L, respectively; thus: $x_\mu^T = x_\mu - x.\hat{P}\hat{P}_\mu$, $x_L = x.\hat{P}$, $\hat{P}_\mu = P_\mu/\sqrt{P^2}$, $P_L = \sqrt{P^2}$, $r = \sqrt{-x^{T2}}$.

With these definitions, constraint (3.1) can be written in the following form:

$$C(p)\ \equiv\ 2P_L p_L - (m_1^2 - m_2^2)\ \approx\ 0\ . \tag{3.2}$$

Constraint C allows the elimination of the relative longitudinal momentum p_L and fixes the evolution law with respect to its conjugate variable x_L. The internal dynamics of the system becomes three-dimensional, apart from the spin degrees of freedom, expressed in terms of the transverse vector x^T.

In the presence of constraint C, the individual Klein-Gordon operators become equal:

$$H_0\ \equiv\ (p_1^2 - m_1^2)\Big|_C\ =\ (p_2^2 - m_2^2)\Big|_C\ =\ \frac{P^2}{4} - \frac{1}{2}(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4P^2} + p^{T2}\ . \tag{3.3}$$

The wave equations for two spin-0 particle systems are:

$$(p_1^2 - m_1^2 - \widetilde{V})\widetilde{\Psi}(x_1, x_2) = 0\ , \tag{3.4a}$$

$$(p_2^2 - m_2^2 - \widetilde{V})\widetilde{\Psi}(x_1, x_2) = 0\ , \tag{3.4b}$$

while for spin-$\frac{1}{2}$ fermion-antifermion systems they are [17, 21]:

$$(\gamma_1.p_1 - m_1)\widetilde{\Psi} = (-\gamma_2.p_2 + m_2)\widetilde{V}\widetilde{\Psi}\ , \tag{3.5a}$$

$$(-\gamma_2.p_2 - m_2)\widetilde{\Psi} = (\gamma_1.p_1 + m_1)\widetilde{V}\widetilde{\Psi}\ . \tag{3.5b}$$

[The antifermion Dirac matrices γ_2 act on $\widetilde{\Psi}$, represented as a 4×4 matrix, from the right, in the reverse order of their appearance.] Potentials $\widetilde{V}$ are Poincaré invariant operators and depend on x only through the transverse vector x^T.

Equations (3.4a)-(3.4b) and (3.5a)-(3.5b) can also be written in a unified form. Introducing the individual particle propagators G_1 and G_2 (with i-factors in the numerators in momentum space) and their product G_0 [Eq. (2.14)] and defining

$$\widetilde{g}_0 \;=\; H_0 G_0\Big|_{C(p)} \;, \tag{3.6}$$

Eqs. (3.4a)-(3.4b) and (3.5a)-(3.5b) take the form:

$$(\widetilde{g}_0^{-1} + \widetilde{V})\;\widetilde{\Psi} \;=\; 0 \;. \tag{3.7}$$

In order to establish the connection of constraint theory wave equations and potentials with the Bethe-Salpeter equation and related quantities, one projects, with an appropriate weight factor, which is chosen here to be H_0 [Eq. (3.3)], Green's functions, scattering amplitudes and wave functions on the constraint hypersurface (3.2). Thus, defining the left-projected Green's function $\widetilde{G}$,

$$\widetilde{G}(P,p,p') \;=\; -2i\pi\delta(C(p))H_0 G(P,p,p') \;, \tag{3.8}$$

one can iterate, in the right-hand side, G around $\widetilde{G}$, repeatedly using its integral equation. One ends up with an integral equation satisfied by $\widetilde{G}$, the kernel of which is related to the Bethe-Salpeter kernel K [20]. Defining the constraint theory wave function Ψ_C from the residue of $\widetilde{G}$ at a bound state pole,

$$\Psi_C \;\equiv\; 2\pi 2P_L\delta(C)\widetilde{\Psi} \;=\; -2i\pi\delta(C)H_0\Phi \;, \tag{3.9}$$

one finds that the wave function $\widetilde{\Psi}$ satisfies Eq. (3.7) with $\widetilde{V}$ related to the scattering amplitude by means of a Lippmann-Schwinger-quasipotential type equation [22, 10, 20]:

$$\widetilde{V} \;=\; \widetilde{T}(1-\widetilde{g}_0\widetilde{T})^{-1} \;, \tag{3.10}$$

$$\widetilde{T}(p,p') \;=\; \frac{i}{2P_L}T(p,p')\Big|_{C(p),C(p')} \;. \tag{3.11}$$

[In T, the total four-momentum conservation factor $(2\pi)^4\delta^4(P-P')$ has been amputated.]

Conversely, the Bethe-Salpeter wave function Φ can be reconstructed from Ψ_C with the equation

$$\Phi \;=\; G_0 T(1-\widetilde{g}T)^{-1}\Psi_C \;, \qquad \widetilde{g} \;=\; 2i\pi\delta(C)\widetilde{g}_0 \;, \tag{3.12}$$

G_0 and $\widetilde{g}_0$ being defined in Eqs. (2.14) and (3.6).

Equation (3.10) is the basis for the calculation of the potential of constraint theory from the scattering amplitude. Iterating Eq. (3.10) with respect to $\widetilde{T}$, one finds that $\widetilde{V}$ receives contributions, in addition to those of the ordinary Feynman diagrams, from new diagrams having at least one three-dimensional box sub-diagram, corresponding to the presence of the constraint factor $\widetilde{g}_0$. These diagrams, which we call "constraint diagrams", play a crucial role in the cancellation mechanism of spurious infra-red singularities [19, 20] and in the reorganiszation of the perturbation series.

The infinitesimal gauge transformation law of the constraint theory wave function is obtained by starting from Eqs. (3.9) and using Eqs. (3.12) and (2.7). One has:

$$\begin{aligned}
\delta_\xi \widetilde{\Psi} &= -\frac{i}{2P_L} H_0 \delta_\xi \Phi \Big|_C \\
&= -\frac{i}{2P_L} ie^2 \delta\xi H_0 (\Delta\Phi) \Big|_C \\
&= -\frac{i}{2P_L} ie^2 \delta\xi H_0 \ (\Delta G_0 T) \ (1 - \widetilde{g}_0 \widetilde{T})^{-1} \widetilde{\Psi} \ ,
\end{aligned} \tag{3.13}$$

where the integration inside the term $(\Delta G_0 T)$ is four-dimensional.

The transformation law of the potential is obtained by starting from the relationship between the wave equation operator $(\widetilde{g}_0^{-1} + \widetilde{V})$ [Eq. (3.7)] and the scattering amplitude $\widetilde{T}$ [Eqs. (3.10)-(3.11)]:

$$(\widetilde{g}_0^{-1} + \widetilde{V}) \ = \ \widetilde{g}_0^{-1}(1 - \widetilde{g}_0 \widetilde{T})^{-1} \ = \ (1 - \widetilde{T}\widetilde{g}_0)^{-1} \widetilde{g}_0^{-1} \ . \tag{3.14}$$

One finds the relation

$$\delta_\xi (\widetilde{g}_0^{-1} + \widetilde{V}) \ = \ (\widetilde{g}_0^{-1} + \widetilde{V}) \widetilde{g}_0 (\delta_\xi \widetilde{T}) \widetilde{g}_0 (\widetilde{g}_0^{-1} + \widetilde{V}) \ , \tag{3.15}$$

which, according to the transformation law (2.15), becomes:

$$\begin{aligned}
\delta_\xi (\widetilde{g}_0^{-1} + \widetilde{V}) = \frac{i}{2P_L} ie^2 \delta\xi (\widetilde{g}_0^{-1} + \widetilde{V}) \widetilde{g}_0 \Big\{ & G_0^{-1} (\Delta + \Delta G_0 T) \\
& + (\Delta + T G_0 \Delta) G_0^{-1} - (\text{crossed}) \Big\} \widetilde{g}_0 (\widetilde{g}_0^{-1} + \widetilde{V}) \ .
\end{aligned} \tag{3.16}$$

When this equation is applied on the wave function $\widetilde{\Psi}$, the operator $(\widetilde{g}_0^{-1} + \widetilde{V})$ of the utmost right gives zero [Eq. (3.7)] provided it is not multiplied by terms having the bound state pole at the same position. Therfore, the crossed terms, as well as the single Δ terms, disappear from the equation. Furthermore, in the second term of the right-hand

side of the equation the product $(i/(2P_L))(\widetilde{g}_0^{-1} + \widetilde{V})\widetilde{g}_0 T$ is , according to Eqs. (3.14), the four-dimensional continuation (in relative longitudinal momentum) of $(1 - \widetilde{T}\widetilde{g}_0)^{-1}\widetilde{T} = \widetilde{V}$, which does not have a pole; hence, it does not contribute when Eq. (3.16) is applied on $\widetilde{\Psi}$. One thus obtains the equation:

$$[\delta_\xi(\widetilde{g}_0^{-1} + \widetilde{V})]\ \widetilde{\Psi}\ =\ \frac{i}{2P_L} ie^2 \delta\xi(\widetilde{g}_0^{-1} + \widetilde{V}) H_0(\Delta G_0 T)(1 - \widetilde{g}_0\widetilde{T})^{-1}\ \widetilde{\Psi}\ , \tag{3.17}$$

where Eq. (3.6) was used.

The combination of the two transformation laws (3.13) and (3.17) leads to the weak invariance of the wave equation (3.7):

$$\delta_\xi[\ (\widetilde{g}_0^{-1} + \widetilde{V})\widetilde{\Psi}\ ]\ \approx\ 0\ . \tag{3.18}$$

The invariance of the norm of $\widetilde{\Psi}$ can be shown in a similar way as for the Bethe-Salpeter wave function [Eqs. (2.16)-(2.18)]. The norm of the internal part of $\widetilde{\Psi}$ is [10, 17, 20]:

$$(\widetilde{\psi}, \widetilde{\psi})\ =\ -\eta \int d^3x^T 4P^2\ \overline{\widetilde{\Psi}}\ \frac{\partial}{\partial s}[\widetilde{g}_0^{-1} + \widetilde{V}]\ \widetilde{\psi}\ =\ 2P_L\ , \tag{3.19}$$

where $\overline{\widetilde{\psi}}$ is the adjoint of $\widetilde{\psi}$, equal to $\widetilde{\psi}^*$ in the bosonic case and to $[\gamma_{1L}\gamma_{2L}\widetilde{\psi}]^\dagger$ in the fermionic case; $s = P^2$ and η was defined after Eq. (2.6). Using the transformation laws (3.13) and (3.16) and arguments similar to those used in Eqs. (2.16)-(2.18) one shows the (weak) invariance of the norm (3.19):

$$\delta_\xi(\widetilde{\psi}, \widetilde{\psi})\ =\ 0\ . \tag{3.20}$$

Contrary to the four-dimensional case, the gauge transformation laws (3.13) and (3.17) of the three-dimensional theory explicitly depend on the interaction, a feature that renders their evaluations rather tricky. Furthermore, the amplitude T that appears in the transformation is not submitted on its left to the constraint (3.2). Therefore, the quantity $(i/(2P_L))T(1 - \widetilde{g}_0\widetilde{T})^{-1}$ represents a four-dimensional continuation of potential $\widetilde{V}$ [Eq. (3.10)]. Its evaluation, in the general case, cannot be done in compact form. However, when the local approximation is used for the potentials, simplifications occur. This case is considered in Sec. 4.

4 Gauge transformations in the local approximation of potentials

In lowest order of perturbation theory (one photon exchange), relationship (3.10) provides a local expression for $\widetilde{V}$ in three-dimensional x-space (with respect to x^T). It turns out that this property can also be maintained, in a certain approximation, in higher orders. The perturbation theory calculations effected in Ref. [20] have shown that, in the Feynman gauge, the infra-red leading part of the contribution of the n-photon exchange diagrams can be represented in (three-dimensional) x-space by a local function of r ($= \sqrt{-x^{T2}}$), of the type $(\alpha/r)^n$, α being the fine structure constant. The sum of these leading terms also yields a local function for $\widetilde{V}$. Therefore, the local approximation of $\widetilde{V}$ can be considered as a sensible one: it includes not only lowest order effects, but also leading effects of multiphoton exchange diagrams. The use of this approximation considerably simplifies the resolution of wave equations, where now standard methods of quantum mechanics can be applied. In the rest of this article we shall limit ourselves to this approximation and shall correspondingly consider the transformation laws obtained in Sec. 3.

In the fermionic case, to ensure positivity of the norm, potential $\widetilde{V}$ must satisfy an inequality [21]; for local potentials that commute with $\gamma_{1L}\gamma_{2L}$, the parametrization [23]

$$\widetilde{V} \ = \ \tanh V \ , \tag{4.1}$$

satisfies this condition. Introducing the wave function transformation

$$\widetilde{\Psi} \ = \ (\cosh V)\Psi \ , \tag{4.2}$$

the norm of the internal part of the new wave function Ψ becomes (in the c.m. frame):

$$\int d^3\mathbf{x}\ Tr\left\{\psi^\dagger\Big[1 + 4\gamma_{10}\gamma_{20}P_0^2\frac{\partial V}{\partial P^2}\Big]\psi\right\} \ = \ 2P_0 \ , \tag{4.3}$$

which, for c.m. energy independent potentials, reduces to the conventional free norm of states.

Considering local potentials V (in x^T) composed of combinations of scalar, pseudoscalar and vector potentials,

$$V \ = \ V_1 + \gamma_{15}\gamma_{25}V_3 + \gamma_1^\mu\gamma_2^\nu(g_{\mu\nu}^{LL}V_2 + g_{\mu\nu}^{TT}U_4 + \frac{x_\mu^T x_\nu^T}{x^{T2}}T_4) \ , \tag{4.4}$$

and using transformations (4.1)-(4.2), Eqs. (3.5a)-(3.5b) can be brought into forms analogous to the Dirac equation, where each particle appears as placed in the external potential created by the other particle. The wave equation satisfied by particle 1 becomes [21]:

$$\Big\{\Big[\frac{P_L}{2}e^{2V_2}+\frac{(m_1^2-m_2^2)}{2P_L}e^{-2V_2}\ \Big]\ \gamma_{1L}-\frac{M}{2}e^{2V_1}-\frac{(m_1^2-m_2^2)}{2M}e^{-2V_1}$$
$$+e^{-2U_4}\Bigg[\gamma_1^T.p^T+\frac{i\hbar}{2x^{T2}}(e^{-2T_4}-1)\ (2\gamma_1^T.x^T+i\gamma_1^{T\alpha}\sigma_{2\alpha\beta}^{TT}x^{T\beta})+(e^{-2T_4}-1)\frac{\gamma_1^T.x^T}{x^{T2}}x^T.p^T$$
$$-2i\hbar e^{-2T_4}\gamma_2^T.x^T\Big(\dot V_1+\gamma_{1L}\gamma_{2L}\dot V_2+\gamma_{15}\gamma_{25}\dot V_3+\gamma_1^T.\gamma_2^T\dot U_4+\frac{\gamma_1^T.x^T\gamma_2^T.x^T}{x^{T2}}\dot T_4\Big)\Bigg]\Bigg\}\Psi\ =\ 0\ , \tag{4.5}$$

where we have defined:

$$\dot F\ \equiv\ \frac{\partial F}{\partial x^{T2}}\ ,\qquad \sigma_{a\mu\nu}=\frac{1}{2i}\Big[\gamma_{a\mu},\gamma_{a\nu}\Big]\quad (a=1,2)\ . \tag{4.6}$$

The wave equation satisfied by particle 2 can be obtained from Eq. (4.5) by the replacements : $p_1\leftrightarrow -p_2,\ x\to x,\ m_1\leftrightarrow m_2,\ \gamma_1\leftrightarrow\gamma_2$. We recognize that the scalar potential V_1 acts as a modification of the total mass M of the fermions through the change $M\to Me^{2V_1}$ while $(m_1^2-m_2^2)$ is kept fixed. The timelike vector potential V_2 acts as a modification of the c.m. total energy P_L through the change $P_L\to P_Le^{2V_2}$, while $(p_{1L}^2-p_{2L}^2)=(m_1^2-m_2^2)$ is kept fixed. The spacelike potential U_4 changes the orbital angular momentum operator from $\mathbf{L}$ to $\mathbf{L}e^{-2U_4}$ (in the c.m. frame) and the combination U_4+T_4 of the spacelike potentials changes the radial momentum operator from p_r to $p_re^{-2(U_4+T_4)}$ (in the classical limit). The pseudoscalar potential appears only in spin- and $\hbar$-dependent terms.

In QED, the summation of leading infra-red effects of multiphoton exchange diagrams, as described above [20], leads to the following expressions for the timelike (V_2) and spacelike (U_4 and T_4) parts of the electromagnetic potential in the Feynman gauge:

$$V_2=\frac{1}{4}\ln\Big(1+\frac{2\alpha}{P_Lr}\Big)\ , \tag{4.7}$$
$$U_4=V_2\ ,\qquad T_4\ =\ 0\ . \tag{4.8}$$

Potentials (4.7)-(4.8) are compatible with the minimal substitution rules proposed long ago by Todorov for spin-0 particles, on the basis of an identification of the two-particle

motion in the c.m. frame to that of a fictitious particle with appropriately defined reduced mass and energy [18].These rules were extended to the fermionic case by Crater and Van Alstine [16]. The above potentials were shown to reproduce the correct $O(\alpha^4)$ effects in muonium and positronium spectra [24, 21].

Similar results as above can also be derived with scalar photons contributing to the scalar potential V_1 [Eq. (4.4)] [16, 20].

The application of the previous summation method of the Feynman diagrams to the intercations of bosons shows, as one naturally expects, that the classical parts of the potentials in the fermionic case (written in the Pauli-Schrödinger form [21]) and in the bosonic case coincide [20]. Therefore, one can use unified potentials for both cases.

It is not straightforward to generalize the above evaluation and summation techniques to other covariant gauges than the Feynman gauge. The presence of the additional gauge piece in the photon propagator breaks the permutational symmetry used in the previous calculations and renders their evaluation rather complicated. This is why these potentials will be evaluated from the Feynman gauge, using the infinitesimal transformation laws obtained in Sec. 3.

To have a rough idea of the expected results, we first consider in some detail the one-photon exchange approximation . The corresponding potential is then local, without further approximation. Indeed, Eq. (3.10), specialized to the one-photon exchange diagram, projects, with the constraint condition (3.2), the photon propagator in momentum space on the surface $k_L = 0$; the potential in x-space is obtained with the three-dimensional Fourier transformation with respect to k^T. One finds for the photon propagator in three-dimensional x-space, in the gauge ξ, the expression:

$$\widetilde{D}_{\mu\nu}(x^T) \; = \; \frac{i}{4\pi}\left(g_{\mu\nu}^{LL} + g_{\mu\nu}^{TT}(1-\frac{\xi}{2}) + \frac{x_\mu^T x_\nu^T}{x^{T2}}\frac{\xi}{2}\right)\frac{1}{r} \; ; \tag{4.9}$$

it yields the following expressions for the potentials in the gauge ξ:

$$V_{2\xi} \; = \; \frac{\alpha}{2P_L r} \; , \qquad U_{4\xi} \; = \; V_{2\xi} + U_{g\xi} \; , \qquad T_{4\xi} \; = \; 2x^{T2}\dot{U}_{g\xi} \; , \qquad U_{g\xi} \; = \; -\frac{\xi}{2}\,\frac{\alpha}{2P_L r} \; . \tag{4.10}$$

[The dot operation is defined in Eq. (4.6).] To this order, $V_{2\xi}$ is independent of ξ.

After replacing these potentials in the wave equation (4.5) (and in the equivalent one of particle 2) and designating by Ψ_ξ the corresponding wave function, it can be seen that the wave function transformation

$$\Psi_\xi = U(\xi)\Psi \; , \qquad U(\xi) \; \simeq \; (1 + ie^2\xi S_0) \; ,$$

$$S_0=\frac{1}{2}(Fx^T.p^T+p^T.x^TF)\ ,\qquad F=\frac{1}{4\pi}\,\frac{1}{2P_Lr}\ , \tag{4.11}$$

removes, to first order in α, all the ξ-dependent terms from the wave equation and gives back the wave equation in the Feynman gauge. The operator S_0 can also be written in the following form:

$$S_0\ =\ -\frac{i}{4}\left[H_0,\int^{x^{T2}}dx^{T2}F\right]\ =\ -\frac{i}{2P_L}\left[H_0,\widetilde{\Delta}(x^T)\right]\ , \tag{4.12}$$

where $\widetilde{\Delta}(x^T)$ is the three-dimensionally reduced expression of the gauge propagator $\Delta(x)$ [Eq. (2.3)] (calculated by the Schwinger parametrization and dimensional regularization):

$$\widetilde{\Delta}(x^T)\ =\ \int\frac{d^3k^T}{(2\pi)^3}\,\frac{e^{ik^T.x^T}}{(k^{T2}+i\epsilon)^2}\ =\ -\frac{1}{8\pi}r\ . \tag{4.13}$$

The above study can also be repeated for the bosonic case, the same results as in Eqs. (4.11)-(4.12) being found. [The wave equations for vector interactions with bosons can be found in Refs. [16, 17]. In the first paper of Ref. [17], the eigenvalue equation in the Feynman gauge is given by Eq. (5.12) with the identifications $(1-B)=(1-A)^{-1}=e^{2V_2}=e^{2U_4}$.]

To investigate the transformation laws for the higher order diagrams, we go back to the general case of Eqs. (3.13) and (3.17). We use the wave equation (3.7), together with relation (3.10), in its integral form, valid for a bound state,

$$\widetilde{\Psi}+\widetilde{g}_0\widetilde{T}(1-\widetilde{g}_0\widetilde{T})^{-1}\widetilde{\Psi}\ =\ 0\ , \tag{4.14}$$

and add it, multiplied with an appropriate (nonsingular) factor, to Eq. (3.13). We obtain:

$$\delta_\xi\widetilde{\Psi}\ =\ -\frac{i}{2P_L}ie^2\delta\xi H_0\left\{(\Delta G_0T)+(\widetilde{\Delta}\widetilde{g}_0\widetilde{T})\right\}(1-\widetilde{g}_0\widetilde{T})^{-1}\widetilde{\Psi}-\frac{i}{2P_L}ie^2\delta\xi H_0\widetilde{\Delta}\widetilde{\Psi}\ . \tag{4.15}$$

The quantity $(\widetilde{\Delta}\widetilde{g}_0\widetilde{T})$ is the constraint diagram counterpart of the amplitude (ΔG_0T) [20] and the integration inside it is three-dimensional, after constraint (3.2) is used.

We can still add to Eq. (4.15) the contribution of the crossed diagram counterpart of (ΔG_0T), denoted by $(\Delta G_0T)_{cr}$, in which the gauge propagator Δ crosses the scattering amplitude T (see Fig. 2). This is possible since $(\Delta G_0T)_{cr}$ does not have a pole at the bound state position in the s-channel and hence one can apply the operator $(1-\widetilde{g}_0\widetilde{T})^{-1}$ on $\widetilde{\Psi}$ and obtain zero. Thus, Eq. (4.15) becomes:

$$\begin{aligned}\delta_\xi\widetilde{\Psi}=&-\frac{i}{2P_L}ie^2\delta\xi H_0\left\{(\Delta G_0T)+(\Delta G_0T)_{cr}+(\widetilde{\Delta}\widetilde{g}_0\widetilde{T})\right\}(1-\widetilde{g}_0\widetilde{T})^{-1}\widetilde{\Psi}\\&-\frac{i}{2P_L}ie^2\delta\xi H_0\widetilde{\Delta}\widetilde{\Psi}\ .\end{aligned} \tag{4.16}$$

The sum of the amplitudes $(\Delta G_0 T)$, $(\Delta G_0 T)_{cr}$ and $(\widetilde{\Delta}\widetilde{g}_0\widetilde{T})$ can be evaluated at leading order of the infra-red counting rules of QED with the eikonal approximation [25, 26, 27] adapted to the bound state problem. This approximation was verified to yield the correct results for the leading terms of the two-photon exchange diagrams and then was generalized to higher-order diagrams [20]. It consists of making the following approximations in the fermion propagators:

$$
\begin{aligned}
G_1(p_1-k_1) &\simeq \frac{i}{-2p_1.k_1+i\epsilon}\Big[(\gamma_{1L}p_{1L}+m_1)-\gamma_{1L}k_{1L}\Big]\ ,\\
G_2(-(p_2+k_2)) &\simeq \frac{i}{2p_2.k_2+i\epsilon}\Big[(-\gamma_{2L}p_{2L}+m_2)-\gamma_{2L}k_{2L}\Big]\ ,
\end{aligned}
\tag{4.17}
$$

and of neglecting, at intermediate stages of the calculation, momentum transfers relative to subgraphs of a given graph. Neglecting thus in the amplitude T, in Eq. (4.16), the momentum transfer, the calculation becomes similar to that of two-photon exchange diagrams. (For positivity reasons, we also retain the quadratic term k_L^2 in the product G_1G_2.) One finds:

$$
(\Delta G_0 T)+(\Delta G_0 T)_{cr}+(\widetilde{\Delta}\widetilde{g}_0\widetilde{T})\ \simeq\ \widetilde{\Delta}(2+\frac{H_0}{4p_{1L}p_{2L}})\widetilde{T}\ . \tag{4.18}
$$

At leading order, the term $H_0/(4p_{1L}p_{2L})$ is equivalent to $-\widetilde{g}_0^{-1}$; neglecting quantum effects, the latter can be brought to the utmost right and replaced there by $\widetilde{V}$. One finally obtains:

$$
\delta_\xi\widetilde{\Psi}\ =\ -\frac{i}{2P_L}ie^2\delta\xi H_0\widetilde{\Delta}(1+\widetilde{V}_{lead,\xi})^2\widetilde{\Psi}\ , \tag{4.19}
$$

where $\widetilde{V}_{lead,\xi}$ is the infra-red leading part of $\widetilde{V}$ (in the gauge ξ), i.e., the timelike component of the vector potential and where $\gamma_{1L}\gamma_{2L}$ is replaced by -1:

$$
\widetilde{V}_{lead,\xi}\ =\ -\tanh V_{2\xi}\ . \tag{4.20}
$$

(In the Feynman gauge V_2 is given by Eq. (4.7).)

Similarly, Eq. (3.17) yields:

$$
[\delta_\xi(\widetilde{g}_0^{-1}+\widetilde{V})]\widetilde{\Psi}\ =\ \frac{i}{2P_L}ie^2\delta\xi(\widetilde{g}_0^{-1}+\widetilde{V})H_0\widetilde{\Delta}(1+\widetilde{V}_{lead,\xi})^2\widetilde{\Psi}\ . \tag{4.21}
$$

To integrate Eq. (4.19) up to finite ξ's, we bring the operator H_0 to the utmost right and use the equation of motion (with the approximation $H_0\simeq -4p_{1L}p_{2L}\widetilde{g}_0^{-1}$):

$$
\delta_\xi\widetilde{\Psi}\ =\ -\frac{i}{2P_L}ie^2\delta\xi\Big\{\Big[H_0,\widetilde{\Delta}(1+\widetilde{V}_{lead,\xi})^2\Big]+4p_{1L}p_{2L}\widetilde{\Delta}(1+\widetilde{V}_{lead,\xi})^2\widetilde{V}_{lead,\xi}\Big\}\ \widetilde{\Psi}\ . \tag{4.22}
$$

The solution of this equation is:

$$\begin{aligned}
\widetilde{\Psi}_{\xi_2} &= \overline{U}(\xi_2,\xi_1)\widetilde{\Psi}_{\xi_1} \ , \\
\overline{U}(\xi_2,\xi_1) &= \mathcal{P}\left(\exp\left\{ie^2\int_{\xi_1}^{\xi_2} d\xi W(\xi)\right\}\right) \ , \\
W(\xi) &= -\frac{i}{2P_L}\left\{\left[H_0,\widetilde{\Delta}(1+\widetilde{V}_{lead,\xi})^2\right] + 4p_{1L}p_{2L}\widetilde{\Delta}(1+\widetilde{V}_{lead,\xi})^2\widetilde{V}_{lead,\xi}\right\} \ ,
\end{aligned} \tag{4.23}$$

where $\mathcal{P}$ is the path ordering operator.

In the following, we shall study an approximate form of this transformation law. To this end, we adopt two simplifications. First, we assume that in the commutator, in $W(\xi)$, the potential $\widetilde{V}_{lead,\xi}$ can be approximated by its expression of the Feynman gauge:

$$\widetilde{V}_{lead,\xi} \simeq \widetilde{V}_{lead,F} = \left(\frac{1-\sqrt{1+\frac{2\alpha}{P_L r}}}{1+\sqrt{1+\frac{2\alpha}{P_L r}}}\right) \ . \tag{4.24}$$

Second, we assume that the two operators in $W(\xi)$ are commuting objects. With these approximations the gauge transformation operator $\overline{U}(\xi_2,\xi_1)$ takes the form:

$$\begin{aligned}
\overline{U}(\xi_2,\xi_1) &\simeq T(\xi_2,\xi_1)U(\xi_2-\xi_1) \ , \\
U(\xi) &= e^{ie^2\xi S} \ , \qquad S = -\frac{i}{2P_L}\left[H_0,\widetilde{\Delta}(1+\widetilde{V}_{lead,F})^2\right] \ , \\
T(\xi_2,\xi_1) &= \exp\left\{\frac{e^2}{2P_L}\int_{\xi_1}^{\xi_2} d\xi \ 4p_{1L}p_{2L}\widetilde{\Delta}(1+\widetilde{V}_{lead,\xi})^2\widetilde{V}_{lead,\xi}\right\} \ .
\end{aligned} \tag{4.25}$$

($T(\xi_2,\xi_1)$ and $U(\xi)$ are supposed to be commuting.) As we shall see in Sec. 5, the above approximate forms provide the main qualitative properties of the gauge transformations of wave functions and potentials.

The wave equation operator $(\widetilde{g}_0^{-1}+\widetilde{V})$ transforms as:

$$[\widetilde{g}_0^{-1}+\widetilde{V}]_\xi = T^{-1}(\xi_2,\xi_1)U(\xi_2-\xi_1)(\widetilde{g}_0^{-1}+\widetilde{V})U^\dagger(\xi_2-\xi_1)T^{-1}(\xi_2,\xi_1) \ . \tag{4.26}$$

Transformations (4.25)-(4.26) ensure the (weak) invariance of the norm (3.19).

Among the two transformation operators U and T, it is the former which is the nontrivial one, generating local transformations in x-space, while the latter acts as a multiplicative factor. In the following we shall focus our attention on the properties of the operator U.

5 Properties of gauge transformations in the local approximation of potentials

This section is devoted to the study of the properties of gauge transformations in the local approximation of potentials, implemented by the operator U [Eq. (4.25)]. There is a complete similarity between the cases of bosons and fermions (the operator U is spin independent) and for this reason we shall concentrate on the case of fermions only. The generator S of the transformations will be written in a form similar to that of Eq. (4.11), which is more tractable for practical calculations. Thus, the gauge transformation operator $U(\xi)$ is defined as:

$$\widetilde{\Psi}_\xi \;=\; U(\xi)\widetilde{\Psi}\ , \qquad U(\xi) \;=\; e^{i\xi(fx^T.p^T + p^T.x^T f)/(2\hbar)}\ ,$$
$$f \;=\; \frac{\alpha}{2P_L r}\,\frac{\partial}{\partial r}[r(1+\widetilde{V}_{lead,F})^2]\ , \qquad \widetilde{V}_{lead,F} \;=\; \left(\frac{1-\sqrt{1+\frac{2\alpha}{P_L r}}}{1+\sqrt{1+\frac{2\alpha}{P_L r}}}\right)\ . \tag{5.1}$$

The operator $U(\xi)$ acts through changes of the variables x^T and p^T. We denote by $x^T(\xi)$, $r(\xi)$ and $p^T(\xi)$ the new expressions obtained from x^T, r and p^T, respectively, after $U(\xi)$ has acted on them. We have:

$$x^T_\alpha(\xi) \;=\; U(\xi)x^T_\alpha U^\dagger(\xi)\ , \qquad r(\xi) \;=\; U(\xi)rU^\dagger(\xi)\ , \qquad r \;=\; \sqrt{-x^{T2}}\ , \tag{5.2}$$

from which we deduce the differential equations:

$$\frac{\partial x^T_\alpha(\xi)}{\partial \xi} \;=\; -f(r(\xi))x^T_\alpha(\xi)\ , \qquad \frac{\partial r(\xi)}{\partial \xi} \;=\; -f(r(\xi))r(\xi)\ . \tag{5.3}$$

We notice that the variable x^T_α/r remains unchanged under the action of $U(\xi)$, which acts as a local dilatation operator in x-space, and hence it is sufficient to study the variation of r.

The solution of Eq. (5.3) is:

$$\int_r^{r(\xi)} \frac{dz}{zf(z)} \;=\; -\xi\ , \tag{5.4}$$

from which we also deduce:

$$\frac{\partial r(\xi)}{\partial r} \;=\; \frac{r(\xi)f(r(\xi))}{rf(r)}\ . \tag{5.5}$$

The action of $U(\xi)$ on the momentum operator p^T is more involved. The operator $p^T(\xi)$ is no longer parallel to p^T and has components along x^T. For reasons that will

become evident below, we parametrize $p^T(\xi)$ by means of two functions $U_{g\xi} = U_{g\xi}(r(\xi), r)$ and $T_{g\xi} = T_{g\xi}(r(\xi), r)$ as follows:

$$\begin{aligned} p_\alpha^T(\xi) &= U(\xi) p_\alpha^T U^\dagger(\xi) \\ &= e^{-2U_{g\xi}} p_\alpha^T + e^{-2U_{g\xi}} \left(e^{-2T_{g\xi}} - 1 \right) \frac{x_\alpha^T}{x^{T2}} x^T.p^T \\ &\quad + i\hbar x_\alpha^T \left[\frac{1}{x^{T2}} e^{-2U_{g\xi}} \left(e^{-2T_{g\xi}} - 1 \right) - 2(\dot{U}_{g\xi} + \dot{T}_{g\xi}) e^{-2(U_{g\xi} + T_{g\xi})} \right] \ . \end{aligned} \tag{5.6}$$

[The dot operation is defined in Eq. (4.6).] The last term, proportional to $i\hbar x_\alpha^T$, is fixed by the hermiticity condition. From the definition of $p^T(\xi)$ we obtain the differential equation:

$$\begin{aligned} \frac{\partial p_\alpha^T(\xi)}{\partial \xi} &= f(r(\xi)) p_\alpha^T(\xi) + 2\dot{f}(r(\xi)) x_\alpha^T(\xi) x^T(\xi).p^T(\xi) \\ &\quad + 5i\hbar x_\alpha^T(\xi) \dot{f}(r(\xi)) + 2i\hbar x^{T2}(\xi) x_\alpha^T(\xi) \ddot{f}(r(\xi)) \ . \end{aligned} \tag{5.7}$$

[The dot derivations are with respect to $x^{T2}(\xi)$.] Use in both sides of Eq. (5.7) of parametrization (5.6) leads to differential equations concerning the functions $U_{g\xi}$ and $T_{g\xi}$:

$$-2\frac{\partial U_{g\xi}}{\partial \xi} = f(r(\xi)) \ , \qquad -2\frac{\partial T_{g\xi}}{\partial \xi} = 2x^{T2}(\xi) \dot{f}(r(\xi)) \ . \tag{5.8}$$

[The terms proportional to $i\hbar x_\alpha^T$ do not lead to new conditions.] Taking into account Eq. (5.3) and the boundary condition $p^T(\xi = 0) = p^T$, the solutions of Eqs. (5.8) are:

$$U_{g\xi} = \frac{1}{2} \ln \left(\frac{r(\xi)}{r} \right) \ , \qquad T_{g\xi} = \frac{1}{2} \ln \left(\frac{f(r(\xi))}{f(r)} \right) \ . \tag{5.9}$$

These also imply the relation:

$$4r^2 \frac{\partial U_{g\xi}}{\partial r^2} = e^{2T_{g\xi}} - 1 \ . \tag{5.10}$$

In the nonrelativistic limit one has the behaviors

$$U_{g\xi} = \frac{1}{2M} U_{g\xi}^{NR} + O\left(\frac{1}{M^2} \right) \ , \qquad T_{g\xi} = \frac{1}{2M} T_{g\xi}^{NR} + O\left(\frac{1}{M^2} \right) \ . \tag{5.11}$$

In this limit, Eq. (5.10) reduces to the relation:

$$T_{g\xi}^{NR} = 2r^2 \frac{\partial U_{g\xi}^{NR}}{\partial r^2} \ . \tag{5.12}$$

We next study the action of the operator U on the wave equation operator $(\widetilde{g}_0^{-1} + \widetilde{V})$. We first consider the operator $\widetilde{g}_0^{-1}$ [Eq. (3.6)], which is composed of the Dirac operators

$(\gamma_1.p_1 \mp m_1)$ and $(-\gamma_2.p_2 \mp m_2)$. The Dirac operator $(\gamma_1.p_1 - m_1)$, say, becomes $(\gamma_1.p_1 - m_1)_\xi$, where only the operator p^T has changed, according to the transformation law (5.6). The operator $(\gamma_1.p_1 - m_1)_\xi$ has the same structure as the wave equation operator (4.5), in which $V_1 = V_2 = V_3 = 0$ and $U_4 = U_{g\xi}$, $T_4 = T_{g\xi}$. [The terms proportional to the matrices $\sigma_{2\alpha\beta}^{TT}$, present in Eq. (4.5), mutually cancel out when expressions (5.9) are used for $U_{g\xi}$ and $T_{g\xi}$.] A similar conclusion is also obtained with the Dirac operator $(-\gamma_2.p_2 - m_2)_\xi$. The two operators $(\gamma_1.p_1)_\xi$ and $(-\gamma_2.p_2)_\xi$ (strongly) commute and hence $(\widetilde{g}_0^{-1})_\xi$ is a well defined operator, in which the ordering of the Dirac operators is irrelevant.

The action of the operator U on the potential $\widetilde{V}$ is obtained by the replacement in it of r by $r(\xi)$, according to the transformations (5.2) and (5.4):

$$\widetilde{V}_\xi \ = \ \widetilde{V}(r(\xi)) \ . \tag{5.13}$$

Examining then the norm of the wave function $\widetilde{\Psi}_\xi$ [Eq. (3.19)] (in the kernel of which, after the evaluation of the action of $\frac{\partial}{\partial s}$, one uses the equations of motion), one deduces that the passage to the wave function Ψ_ξ, characterized by a norm of the type (4.3), is again obtained with transformations of the type (4.1)-(4.2):

$$\widetilde{V}_\xi \ = \ \tanh V_\xi \ , \qquad \widetilde{\Psi}_\xi \ = \ (\cosh V_\xi)\Psi_\xi \ . \tag{5.14}$$

The Dirac type wave equations satisfied by Ψ_ξ have the same structure as Eqs. (4.5), in which, however, the potentials $U_{g\xi}$ and $T_{g\xi}$ have been added up to the existing potentials $U_4(r(\xi))$ and $T_4(r(\xi))$ of V_ξ. This feature indicates us that these wave equations could also have been obtained from the following wave equation satisfied by a wave function $\widetilde{\Psi}'_\xi$ defined below:

$$(\widetilde{g}_0^{-1} + \widetilde{V}'_\xi)\ \widetilde{\Psi}'_\xi \ = \ 0 \ , \tag{5.15}$$

where $\widetilde{V}'_\xi$ is defined as:

$$\widetilde{V}'_\xi \ = \ \tanh V'_\xi \ , \tag{5.16}$$

and V'_ξ has the following timelike ($V'_{2\xi}$) and spacelike ($U'_{4\xi}$, $T'_{4\xi}$) components:

$$\begin{aligned} V'_{2\xi} \ &= \ V_{2\xi} \ = \ V_2(r(\xi)) \ , \\ U'_{4\xi} \ &= \ U_{4\xi} + U_{g\xi} \ = \ U_4(r(\xi)) + U_{g\xi}(r(\xi), r) \ , \\ T'_{4\xi} \ &= \ T_{4\xi} + T_{g\xi} \ = \ T_4(r(\xi)) + T_{g\xi}(r(\xi), r) \ . \end{aligned} \tag{5.17}$$

The wave function $\widetilde{\Psi}'_\xi$ is related to Ψ_ξ by the transformation:

$$\widetilde{\Psi}'_\xi \;=\; (\cosh V'_\xi)\Psi_\xi \; . \tag{5.18}$$

The relationship between $\widetilde{\Psi}'_\xi$ and $\widetilde{\Psi}_\xi$ is:

$$\widetilde{\Psi}'_\xi \;=\; (\cosh V'_\xi)(\cosh V_\xi)^{-1}\widetilde{\Psi}_\xi \; . \tag{5.19}$$

Therefore, the two wave equation operators $[\widetilde{g}_0^{-1}+\widetilde{V}]_\xi$ and $(\widetilde{g}_0^{-1}+\widetilde{V}'_\xi)$ are equivalent:

$$[\widetilde{g}_0^{-1}+\widetilde{V}]_\xi \;\approx\; \widetilde{g}_0^{-1}+\widetilde{V}'_\xi \; . \tag{5.20}$$

The advantage of the representation $\widetilde{\Psi}'_\xi$ is that its wave equation operator has the conventional form (3.7) with a potential $\widetilde{V}'_\xi$ which is local. In this representation, among the three potentials V_2, U_4 and T_4, only V_2 has a form invariant transformation law. If the scalar and pseudoscalar potentials, V_1 and V_3, were present, they would transform as V_2. (In this case $\widetilde{V}_{lead}$ [Eqs. (5.1)] should also contain the scalar potential V_1.)

The above transformation laws satisfy the group property, as can be checked by composing two succesive transformations; hence, they can be used starting from any gauge.

Let us now return to the explicit expression of the function f, Eq. (5.1). Equation (5.4) then yields:

$$\frac{1}{(2x(\xi)-1)}\exp\left(\frac{2}{x(\xi)-1}\right) \;=\; \frac{1}{(2x-1)}\exp\left(\frac{2}{x-1}-\xi\right) \; , \qquad x \;=\; \sqrt{1+\frac{2\alpha}{P_L r}} \; . \tag{5.21}$$

It does not seem possible to express $r(\xi)$ in compact form in terms of r and ξ. However, the above equation provides easily the asymptotic behaviors of $r(\xi)$:

$$r(\xi) \underset{r\to\infty}{=} r - \frac{\alpha\xi}{2P_L} + O\left(\frac{1}{r}\right) \; , \tag{5.22a}$$

$$r(\xi) \underset{r\to 0}{=} re^{-2\xi}\left(\frac{1+\sqrt{\frac{P_L r}{2\alpha}}}{1+\sqrt{\frac{P_L r}{2\alpha}}e^{-\xi}}\right)\exp\left\{4\sqrt{\frac{P_L r}{2\alpha}}(1-e^{-\xi})\right\} + O(r^2) \;\simeq\; re^{-2\xi} \; . \tag{5.22b}$$

We have plotted, in Fig. 3, the curves $r(\xi)$ (in units of $2\alpha/P_L$) for three values of the gauge parameter, $\xi=-2$ (Yennie gauge), $\xi=0$ (Feynman gauge) and $\xi=1$ (Landau gauge). It is observed that the large-distance behavior (5.22a) is reached very rapidly.

From Eqs. (5.1) or Eq. (5.5) one also obtains the asymptotic behaviors of $f(r(\xi))/f(r)$:

$$\frac{f(r(\xi))}{f(r)} \underset{r\to\infty}{=} 1 + \frac{\alpha\xi}{2P_L r} + O\left(\frac{1}{r^2}\right) \ , \tag{5.23a}$$

$$\frac{f(r(\xi))}{f(r)} \underset{r\to 0}{=} 1 + \frac{1}{2}\sqrt{\frac{P_L r}{2\alpha}}\left[\frac{1}{1+\sqrt{\frac{P_L r}{2\alpha}}} - \frac{e^{-\xi}}{1+\sqrt{\frac{P_L r}{2\alpha}}e^{-\xi}} + 4(1-e^{-\xi})\right] + O(r) \ . \tag{5.23b}$$

Equations (5.22a)-(5.23b), together with Eqs. (5.9) and (5.17), yield the behaviors of the new potentials in the corresponding limits. The large-distance expansions (5.22a) and (5.23a) are particularly relevant in the nonrelativistic limit.

Of particular interest is the short-distance behavior of the effective potential. According to Eqs. (5.1)-(5.2), the wave function $\widetilde{\Psi}_\xi(x^T)$ is equal to $\widetilde{\Psi}(x^T(\xi))$; Eq. (5.22b) indicates us that $r(\xi)$ behaves like r near the origin and therefore the behavior of the wave function does not change there under the gauge transformation; this in turn means that the dominant short-distance singularity of the effective potential has remained the same. These features can also be verified explicitly from the wave equations.

To this end, let us consider, for the electromagnetic interaction, the Pauli-Schrödinger type eigenvalue equation obtained from the wave equation (4.5) (in the c.m. frame) [21]:

$$\Bigg\{ e^{4(U_4+T_4)}\left[\frac{P^2}{4}e^{4V_2} - \frac{1}{2}(m_1^2+m_2^2) + \frac{(m_1^2-m_2^2)^2}{4P^2}e^{-4V_2}\right] - \mathbf{p^2} - \mathbf{L^2}\frac{1}{r^2}(e^{4T_4}-1) + \cdots \Bigg\}\phi_3 = 0 \ , \tag{5.24}$$

where ϕ_3 is a reduced (four-component) wave function and $\mathbf{L}$ is the orbital angular momentum operator; the dots stand for spin- and $\hbar$-dependent terms, which are not relevant for the present purpose. The dominant short-distance singularity is provided by the term $e^{4(V_2+U_4+T_4)}$. In the Feynman gauge, [Eqs. (4.7)-(4.8)], it yields the singularity α^2/r^2, which is attractive and produces at the critical value $\alpha_c = \frac{1}{2}$ the fall to the center phenomenon. A detailed analysis of this problem has shown that the theory undergoes at this value of α a chiral phase transition [28]; such a conclusion has also been reached from the Bethe-Salpeter equation in the ladder approximation [29] and from lattice theory calculations [30, 31]. Considering Eq. (5.24) in the gauge ξ, the potentials V_2, U_4 and

T_4 become replaced by $V'_{2\xi}$, $U'_{4\xi}$ and $T'_{4\xi}$, respectively. We find that the modification of the coefficient of the short-distance singularity coming from the form invariant part of $V'_{2\xi} + U'_{4\xi}$, i.e., from $V_{2\xi} + U_{4\xi}$, is cancelled by that coming from the form noninvariant part $U_{g\xi} + T_{g\xi}$, and therefore the same singularity α^2/r^2 emerges again.

One consequence of this result is that the critical coupling constant α_c has the same value $\frac{1}{2}$ in all gauges. This is a consistency check of the formalism, since α, representing here the invariant charge, should lead to a gauge invariant critical value α_c. (In the present approximation, where radiative corrections are neglected, the physical and bare charges are identical; in any event, both quantities should be gauge invariant.) This is in contrast with the results obtained from the Bethe-Salpeter equation in the ladder approximation, where the value of α_c is gauge dependent (equal to $\pi/4$ in the Feynman gauge and to $\pi/3$ in the Landau gauge [29, 32]).

We also emphasize the particular role played by $\widetilde{V}_{lead}$, in f [Eq. (5.1)], in the short-distance behavior of the potentials in the gauge ξ. If $\widetilde{V}_{lead}$ were absent, then the exact solution of Eq. (5.4) would be $r(\xi) = r - \alpha\xi/(2P_L)$ (the same as the asymptotic behavior (5.22a)), producing in the Coulomb potential a singularity shifted to the position $r = \alpha\xi/(2P_L)$. On the other hand, when $\widetilde{V}_{lead}$ is present in f, the function rf (cf. Eq. (5.4)) vanishes like r when r tends to 0 and, as a result, the singularity of the Coulomb potential becomes maintained at the position $r = 0$ in the gauge ξ.

The results obtained so far with the approximations (4.25) are not qualitatively modified when the dependence on ξ of $\widetilde{V}_{lead,\xi}$ is introduced with an iterative treatment. This is a consequence of the asymptotic behaviors (5.22a)-(5.22b) and of the fact that the corresponding behaviors of the function $\widetilde{\Delta}(1 + \widetilde{V}_{lead,\xi})^2$ in f [Eq. (5.1)] are not changed. Similarly, one can also estimate the effect of the operator T [Eq. (4.25)]. The integrand in the argument of the exponential in T is a positive function. The behavior of the integral can be studied in the asymptotic regions with the aid of relations (5.22a)-(5.22b). In the limit $r \to 0$, the argument of the exponential tends to zero, while for $r \to \infty$, it tends to the constant value $(\xi_2 - \xi_1)\alpha^2 p_{1L}p_{2L}/(2P_L^2)$ and generally remains a smooth function between these two limits. These properties justify the factorization approximation made at the level of the gauge transformation operator $\overline{U}$ [Eqs. (4.23) and (4.25)].

Let us finally remark that transformations (5.1), because of their spin-independent character, can also be applied in the case of bosonic wave equations, where now the

operator $\widetilde{g}_0^{-1}$ is equal to $-H_0$, and the distinction between the representations $\widetilde{\Psi}_\xi$ and $\widetilde{\Psi}'_\xi$ becomes irrelevant.

6 Summary and concluding remarks

Using connection with quantum field theory, we established the infinitesimal covariant abelian gauge transformation laws of constraint theory two-particle wave functions and potentials and showed weak invariance of the coresponding wave equations. Contrary to the four-dimensional case and because of the three-dimensional projection operation, these transformation laws are interaction dependent.

Simplifications occur when one sticks to local potentials, which result, in each formal order of perturbation theory, from the infra-red leading effects of multiphoton exchange diagrams. In this case, the finite gauge transformation can explicitly be represented, with a suitable approximation and up to a multiplicative factor, by a momentum dependent unitary operator that acts in x-space as a local dilatation operator. The latter acts on the potentials through two kinds of modification: a change of the argument r of the potentials into a function $r(\xi)$ and a functional change of certain parts of the spacelike components of the electromagnetic potential. The function $r(\xi)$ is essentially dominated by its large-distance behavior, in which r is simply shifted by a constant value. The knowledge of these modifications allows one to reconstruct, starting from the Feynman gauge, the potentials in other covariant gauges. It was shown that the dominant short-distance singularity of the effective potential of the Pauli-Schrödinger type eigenvalue equation is gauge invariant with a critical value α_c of the coupling constant equal to 1/2.

The above results allow the search for optimal gauges when incorporating new contributions into the potentials, which might come either from QED, or from other interactions. For instance, it is known that vacuum polarization affects only the transverse part of the photon propagataor. Therefore, the choice of the Landau gauge for the introduction of the effective charge seems to be most indicated.

The similarity in structure between QED and perturbative QCD, up to the color gauge group matrices and the difference in the effective charges, allows us to envisage the consideration of many of the previous results in problems of quarkonium spectroscopy, where one has also to incorporate at large distances the effects coming from the confining potential. Here also the search for optimal gauges may become useful for subsequent applications.

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Figures

Fig. 1. Infinitesimal change, under gauge transformations, of two-particle Green's function. (One-particle radiative corrections are neglected.) The dashed line, including its vertices, is the gauge propagator $ie^2\delta\xi/(k^2)^2$.

Fig. 2. The three diagrams entering in the evaluation of the interaction dependent part of the change of the constraint theory wave function. The shaded box is the off-mass shell scattering amplitude. The cross indicates the constraint diagram.

Fig. 3. The curves $r(\xi)$ versus r (in units of $2\alpha/P_L$) for three values of the gauge parameter, $\xi = -2$ (Yennie gauge), $\xi = 0$ (Feynman gauge) and $\xi = 1$ (Landau gauge).

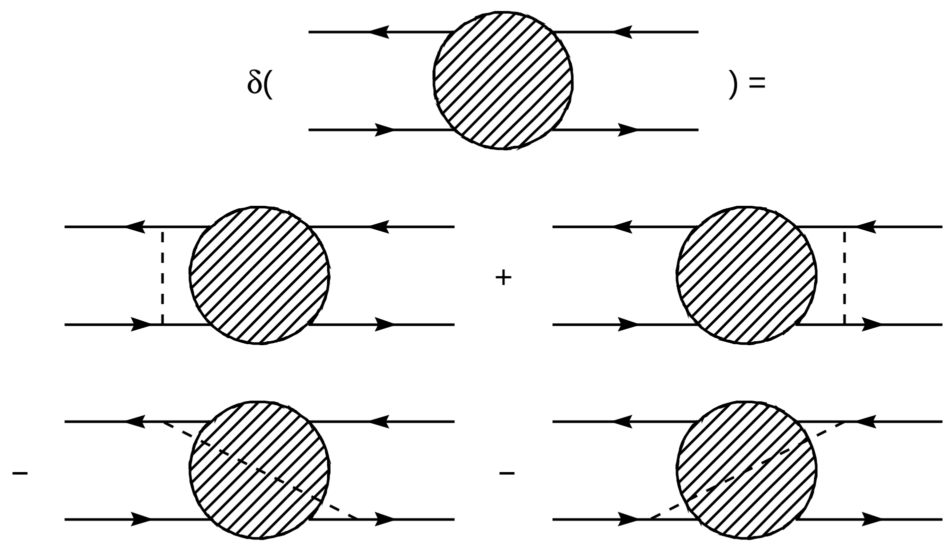

Fig. 1

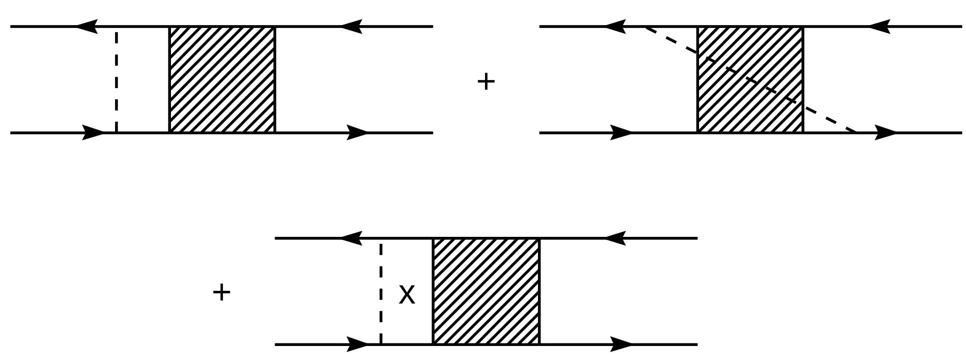

Fig. 2

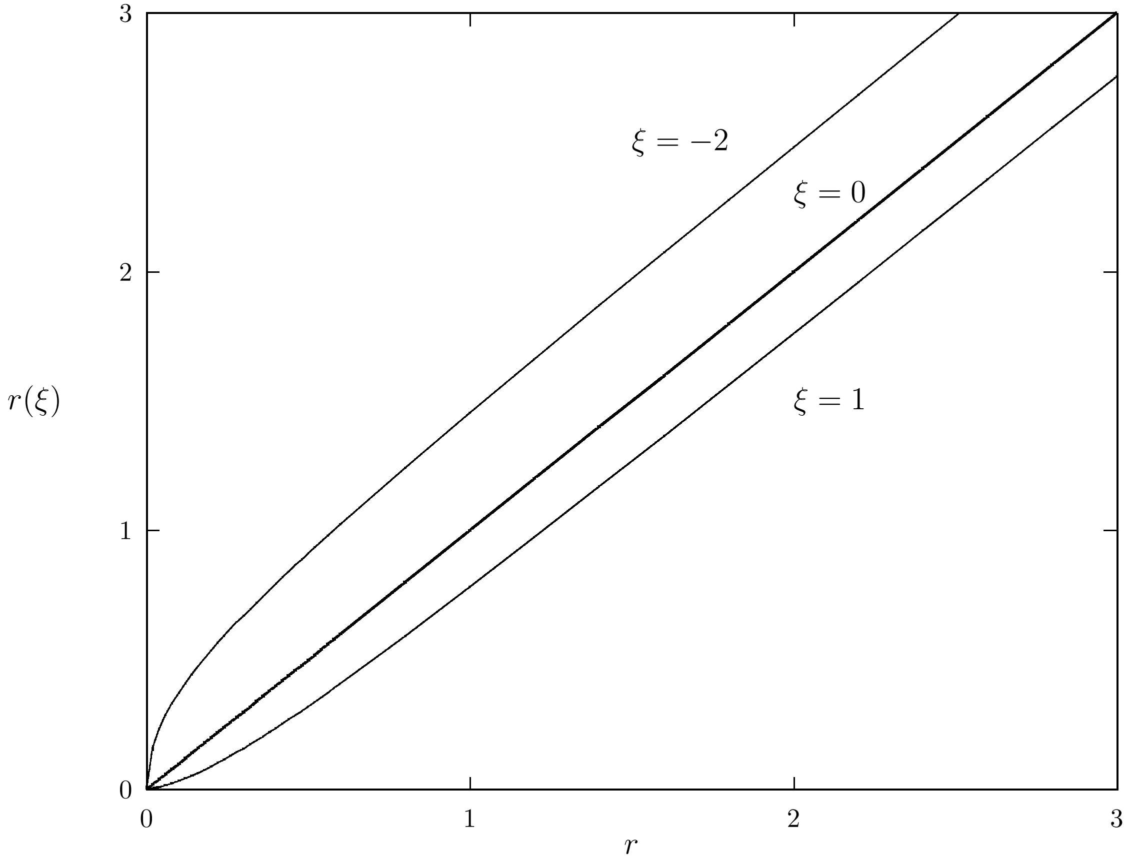

Fig. 3